Extension of level-spacing universality

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(Received 21 February 1997)

In the theory of random matrices, several properties are known to be universal, i.e., independent of the specific probability distribution. For instance, Dyson's short-distance universality of the correlation functions implies the universality of $P(s)$, the level-spacing distribution. We first briefly review how this property is understood for unitary invariant ensembles and consider next a Hamiltonian $H=H_0+V$, in which H_0 is a given, nonrandom, $N\times N$ matrix, and V is an Hermitian random matrix with a Gaussian probability distribution. The standard techniques, based on orthogonal polynomials, which are the key for the understanding of the $H_0=0$ case, are no longer available. Then using a completely different approach, we derive closed expressions for the n-point correlation functions, which are exact for finite N. Remarkably enough the result may still be expressed as a determinant of an $n\times n$ matrix, whose elements are given by a kernel $K(\lambda,\mu)$ as in the $H_0=0$ case. From this representation we can show that Dyson's short-distance universality still holds. We then conclude that $P(s)$ is independent of H_0. [S1063-651X(97)06207-7]

PACS number(s): 05.45.+b, 05.40.+j

I. INTRODUCTION

Many years ago Wigner [1] introduced the level-spacing probability distribution $P(s)$, in his discussion of nuclear energy levels. The exact form of $P(s)$ was found later in the theory of random matrices for the Gaussian unitary ensemble (GUE) [2–4]. This level-spacing probability distribution $P(s)$ was empirically found to be universal in many different cases, for instance, non-Gaussian probability distributions or band matrices (in which case the measure is not unitary invariant), and even for problems of an *a priori* different nature, such as the level spacing of the zeros of the Riemann ζ function [2,5,6], which is known to coincide with that of the GUE.

In Sec. II, we first review how the universality of $P(s)$ has been derived for non-Gaussian unitary invariant ensembles, in which the probability measure is given by

$$P(H)=\frac{1}{Z}e^{-N\,\mathrm{Tr}f(H)}, \tag{1.1}$$

where $f(x)$ is an arbitrary polynomial. One first integrates out the unitary group in order to obtain a probability distribution for the eigenvalues of H. It is then easy to show that the n-point function may be written as an $n\times n$ determinant; the matrix elements of this determinant are given by a kernel expressed in terms of orthogonal polynomials with respect to the weight $\exp[-Nf(x)]$. Then the understanding of the relevant asymptotic behavior of these polynomials at large order allows one to prove the short-distance universality of this kernel. From there one can derive the universality of $P(s)$ in the scaling limit in which N goes to infinity, the distance x between two neighboring eigenvalues goes to zero, and $s=Nx$ is held fixed.

In the third section we consider a Hamiltonian which is the sum of a given deterministic part H_0 and of a random potential V with a Gaussian probability distribution. The measure is not unitary invariant, but one can still write the probability distribution for the eigenvalues of H through the well-known Itzykson-Zuber integral [7]. Generalizing a method introduced by Kazakov [8] for the density of eigenvalues, we write a representation of the n-level correlation function, in terms of an exact and explicit integral over $2n$ variables. Then one discovers that an amazing algebraic structure allows one to express again this n-point function in terms of a determinant of an $n\times n$ matrix. The matrix elements are given by a kernel which has an explicit representation as an integral over two variables. In a previous paper [9], we had already discussed the two-level correlation function of this Hamiltonian through the same method, and we had shown that the behavior of this correlation function is indeed universal, i.e., independent of the Hamiltonian H_0, in the short-range scaling limit, in which the distance x of the two energy levels becomes small, and N goes to infinity, with fixed Nx. We had also briefly discussed the n-point function in [10]. The main steps are recalled here; the universality of $P(s)$ follows immediately.

In Sec. VI we establish some properties of this kernel, and show that it does satisfy some necessary consistency conditions.

II. LEVEL-SPACING DISTRIBUTION $P(S)$ FOR GENERALIZED GUE ENSEMBLES

We return to the single random matrix case with a probability

$$P(H)=\frac{1}{Z}e^{-N\,\mathrm{Tr}f(H)} \tag{2.1}$$

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and integrate out the unitary degrees of freedom. The resulting probability distribution for the N eigenvalues of H is [2]

$$P_N(x_1,\ldots,x_N)=C\prod_{i<j}(x_i-x_j)^2 e^{-N\Sigma_{i=1}^N f(x_i)}. \quad (2.2)$$

The n-point correlation function $R_n(x_1,\ldots,x_n)$, is defined as

$$R_n(x_1,\ldots,x_n)=\frac{N!}{(N-n)!}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_{n+1}\cdots dx_N P_N \times(x_1,\ldots,x_N). \quad (2.3)$$

Following Mehta [2], we introduce the orthogonal polynomials $\phi_k(x)$ with respect to the measure $\exp[-Nf(x)]$. Then R_n is given by the determinant,

$$R_n(x_1,\ldots,x_n)=\det[K_N(x_i,x_j)]_{i,j=1,\ldots,n}. \quad (2.4)$$

in which the kernel $K_N(x,y)$ is expressed as a sum of orthogonal polynomials

$$K_N(x,y)=\frac{1}{N}e^{-(N/2)(f(x)+f(y))}\sum_{k=0}^{N-1}\phi_k(x)\phi_k(y). \quad (2.5)$$

For instance, the pair correlation function, the $n=2$ case, becomes

$$R_2(x_1,x_2)=\rho(x_1)\rho(x_2)-K_N(x_1,x_2)K_N(x_2,x_1) \quad (2.6)$$

in which the density of states $\rho(x)$ is the diagonal part of the kernel $\rho(x)=K_N(x,x)$. With our normalization conventions the density of state $\rho(x)$ has a support of finite extension in the large N limit.

In the short-distance scaling limit, $K_N(x_1,x_2)$ becomes [11–13]

$$K_N(x_1,x_2)\simeq\frac{\sin\left[\pi N(x_1-x_2)\rho\left(\frac{(x_1+x_2)}{2}\right)\right]}{\pi N(x_1-x_2)}, \quad (2.7)$$

for $N\to\infty$, $x_1-x_2\to 0$ and finite $N(x_1-x_2)$. The universality of Eq. (2.7) with respect to the function $f(x)$ which characterizes the probability measure is thus manifest. The universality of the level-spacing distribution $P(s)$ follows at once.

Indeed, following Mehta [2], we first compute the probability $E(\theta)$ that the interval $[-\theta/2,\theta/2]$ does not contain any of the points $x_1,\ldots,x_N$ in the large N limit. It is thus obtained by integrating the N variables of $P_N(x_1,\ldots,x_N)$ outside the interval $[-\theta/2,\theta/2]$:

$$E(\theta)=\int_{out}\cdots\int_{out}P_N(x_1,\ldots,x_N)dx_1\cdots dx_N, \quad (2.8)$$

where the integrals are performed outside the region $[-\theta/2,\theta/2]$;

$$\int_{out}dx=\left(\int_{-\infty}^{\infty}-\int_{-\theta/2}^{\theta/2}\right)dx. \quad (2.9)$$

We may thus express $E(\theta)$ in terms of the R_n's by using systematically Eq. (2.9) for all the N variables

$$E(\theta)=1-N\int_{-\theta/2}^{\theta/2}\rho(x)dx+\frac{N^2}{2!}\int_{-\theta/2}^{\theta/2}\int_{-\theta/2}^{\theta/2}R_2(x,y)dxdy+\cdots. \quad (2.10)$$

The natural scale for the level spacing θ is of order $1/N$ since in the large N limit the support of the density of state is finite. We thus consider the short-distance scaling limit, in which θ goes to zero and N to infinity, with fixed $N\theta$. In that scaling limit

$$N\int_{-\theta/2}^{\theta/2}\rho(x)dx=N\theta\rho(0)+O(1/N). \quad (2.11)$$

We thus define the scaling variable

$$N\theta\rho(0)=s. \quad (2.12)$$

The next terms of Eq. (2.10) are obtained in this limit by the change of variables

$$Nx\rho(0)=x'. \quad (2.13)$$

Then, in the scaling limit,

$$N^2\int_{-\theta/2}^{\theta/2}\int_{-\theta/2}^{\theta/2}R_2(x,y)dx\,dy=\int_{-s/2}^{s/2}\int_{-s/2}^{s/2}\tilde{R}_2(x',y')dx'dy', \quad (2.14)$$

with

$$\tilde{R}_n(x_1,\ldots,x_n)=\det[\tilde{K}(x_i,x_j)]_{i,j=1,\ldots,n}, \quad (2.15)$$

in which

$$\tilde{K}(y_1,y_2)=\frac{\sin[\pi(y_1-y_2)]}{\pi(y_1-y_2)}. \quad (2.16)$$

In this scaling limit we thus obtain

$$E(s)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_{-s/2}^{s/2}\cdots\int_{-s/2}^{s/2}dx_1,\ldots,dx_n \times\det[\tilde{K}(x_i,x_j)]_{i,j=1,\ldots,n}. \quad (2.17)$$

From this representation it is easy to expand $E(s)$ for small s; for instance,

$$\int_{-s/2}^{s/2}\int_{-s/2}^{s/2}\tilde{R}_2(x,y)dx\,dy=\frac{\pi^2}{36}s^4-\frac{\pi^4}{675}s^6+O(s^8), \quad (2.18)$$

and since the $n=3$ term of Eq. (2.17) is easily shown to be of order s^7 for small s, we find

$$E(s)=1-s+\frac{\pi^2}{36}s^4-\frac{\pi^4}{675}s^6+O(s^7). \quad (2.19)$$

One can also introduce the eigenvalues $\lambda_i(s)$ of the integral equation for the kernel $\tilde{K}$ on the interval $[-s/2,+s/2]$,

$$\int_{-s/2}^{s/2}\widetilde{K}(x,y)\psi_i(y)dy=\lambda_i\psi_i(x). \tag{2.20}$$

From Eq. (2.17) we can write

$$E(s)=\prod_{i=1}^{\infty}(1-\lambda_i)=\det[1-\widetilde{K}]. \tag{2.21}$$

For small s, a perturbational expansion using Legendre polynomial gives the same result as Eq. (2.19). The level-spacing probability distribution $P(s)$ is now obtained from $E(s)$

$$P(s)=\frac{d^2}{ds^2}E(s). \tag{2.22}$$

Through this representation, we find that the universality of $P(s)$ results from two sources; (i) the n-point correlation R_n is expressed as the determinant of a kernel $K_N(x_i,x_j)$, (ii) the kernel $K_N(x,y)$ has a universal short-distance behavior $\widetilde{K}$ in the short-distance scaling limit.

III. DETERMINISTIC PLUS RANDOM HAMILTONIAN

We now consider a Hamiltonian $H=H_0+V$, where H_0 is a given, nonrandom, $N\times N$ Hermitian matrix, and V is a random Gaussian Hermitian matrix. The probability distribution $P(H)$ is thus given by

$$P(H)=\frac{1}{Z}e^{-(N/2)\mathrm{Tr}V^2}$$
$$=\frac{1}{Z'}e^{-(N/2)\mathrm{Tr}(H^2-2H_0H)}. \tag{3.1}$$

We are thus dealing with a Gaussian unitary ensemble modified by the external matrix source H_0, which breaks the unitary invariance of the measure. In previous works [9,10], we have already discussed the density of state, and the two-level correlation function. For completeness, we repeat here the basic steps. The density of state $\rho(\lambda)$ is

$$\rho(\lambda)=\frac{1}{N}\langle\mathrm{Tr}\,\delta(\lambda-H)\rangle$$
$$=\int_{-\infty}^{+\infty}\frac{dt}{2\pi}e^{-iNt\lambda}U(t), \tag{3.2}$$

where $U(t)$ is the average ''evolution'' operator

$$U(t)=\langle\mathrm{Tr}e^{iNtH}\rangle. \tag{3.3}$$

We first integrate over the unitary matrix ω which diagonalizes H in Eq. (3.1), and without loss of generality we may assume that H_0 is a diagonal matrix with eigenvalues $(\epsilon_1,\dots,\epsilon_N)$. This is done with the help of the well-known Itzykson-Zuber integral [7],

$$\int d\omega\exp(\mathrm{Tr}A\omega B\omega^\dagger)=\frac{\det[\exp(a_ib_j)]}{\Delta(A)\Delta(B)}, \tag{3.4}$$

where $\Delta(A)$ is the Van der Monde determinant constructed with the eigenvalues of A

$$\Delta(A)=\prod_{i<j}^{N}(a_i-a_j). \tag{3.5}$$

We are then led to

$$U(t)=\frac{1}{Z'\Delta(H_0)}\sum_{\alpha=1}^{N}\int dx_1\cdots dx_N e^{iNtx_\alpha}\Delta(x_1,\dots,x_N)$$
$$\times\exp\left(-\frac{N}{2}\sum x_i^2+N\sum\epsilon_ix_i\right). \tag{3.6}$$

The normalization is fixed by

$$U(0)=N. \tag{3.7}$$

The integration over the x_i's may be done easily, if we note that

$$\int dx_1\cdots dx_N\Delta(x_1,\dots,x_N)\exp\left(-\frac{N}{2}\sum x_i^2+N\sum b_ix_i\right)$$
$$=\Delta(b_1,\dots,b_N)\exp\left(\frac{N}{2}\sum b_i^2\right). \tag{3.8}$$

Putting $b_i=\epsilon_i+it\delta_{\alpha,i}$, we obtain

$$U(t)=\sum_{\alpha=1}^{N}\prod_{\gamma\neq\alpha}^{N}\left(\frac{\epsilon_\alpha-\epsilon_\gamma+it}{\epsilon_\alpha-\epsilon_\gamma}\right)e^{-(Nt^2/2)+Nit\epsilon_\alpha}. \tag{3.9}$$

The sum over N terms in Eq. (3.9) may then be replaced by a contour integral in the complex plane,

$$U(t)=\frac{1}{it}\oint\frac{du}{2\pi i}\prod_{\gamma=1}^{N}\left(\frac{u-\epsilon_\gamma+it}{u-\epsilon_\gamma}\right)e^{-(Nt^2/2)+itNu}. \tag{3.10}$$

The contour of integration encloses all the eigenvalues ϵ_γ. The Fourier transform with respect to t gives the density of state in the presence of an arbitrary external source H_0 and for finite N.

In the case of the two-point correlation function, we have

$$R_2(\lambda,\mu)=\left\langle\frac{1}{N}\mathrm{Tr}(\lambda-H)\frac{1}{N}\mathrm{Tr}(\mu-H)\right\rangle. \tag{3.11}$$

By using integral representations for the two δ functions, the two-point correlation function $R_2(\lambda,\mu)$ is expressed as the Fourier transform of $U(t_1,t_2)$,

$$U(t_1,t_2)=\langle\mathrm{Tr}e^{iNt_1H}\mathrm{Tr}e^{iNt_2H}\rangle. \tag{3.12}$$

Again using the Itzykson-Zuber formula to integrate over the unitary group, we obtain

$$U(t_1,t_2)=\sum_{\alpha_1,\alpha_2=1}^{N}\int\prod_{i=1}^{N}dx_i\frac{\Delta(x)}{\Delta(H_0)}$$
$$\times e^{-N\Sigma[(1/2)x_i^2-x_i\epsilon_i]+iN(t_1x_{\alpha_1}+t_2x_{\alpha_2})}. \tag{3.13}$$

After integration over the x_i's, we have

$$U(t_1,t_2)=\sum_{\alpha_1,\alpha_2}\frac{\prod_{i<j}[\epsilon_i-\epsilon_j+it_1(\delta_{i,\alpha_1}-\delta_{j,\alpha_1})+it_2(\delta_{i,\alpha_2}-\delta_{j,\alpha_2})]}{\prod_{i<j}(\epsilon_i-\epsilon_j)}e^{Nit_1\epsilon_{\alpha_1}+Nit_2\epsilon_{\alpha_2}-(N/2)t_1^2-(N/2)t_2^2-Nt_1t_2\delta_{\alpha_1,\alpha_2}}. \quad (3.14)$$

This term is devided into two parts; $\alpha_1=\alpha_2$ and $\alpha_1\neq\alpha_2$ cases,

$$U(t_1,t_2)=\sum_{\alpha_1}\prod_{i<j}\frac{[\epsilon_i-\epsilon_j+i(t_1+t_2)(\delta_{i,\alpha_1}-\delta_{j,\alpha_1})]}{(\epsilon_i-\epsilon_j)}e^{Ni(t_1+t_2)\epsilon_{\alpha_1}-(N/2)(t_1+t_2)^2}$$
$$+\sum_{\alpha_1\neq\alpha_2}\frac{(\epsilon_{\alpha_1}-\epsilon_{\alpha_2}+i(t_1-t_2))}{\epsilon_{\alpha_1}-\epsilon_{\alpha_2}}\prod_{\gamma\neq(\alpha_1,\alpha_2)}\frac{(\epsilon_{\alpha_1}-\epsilon_\gamma+it_1)}{\epsilon_{\alpha_1}-\epsilon_\gamma}\frac{(\epsilon_{\alpha_2}-\epsilon_\gamma+it_2)}{\epsilon_{\alpha_2}-\epsilon_\gamma}e^{Nit_1\epsilon_{\alpha_1}+Nit_2\epsilon_{\alpha_2}-(N/2)(t_1^2+t_2^2)}. \quad (3.15)$$

Fourier transform of the first term becomes δ function [14], and can be neglected for $R_2(\lambda,\mu)$ for $\lambda\neq\mu$. The double sum in Eq. (3.15) may be written again as an integral over two complex variables

$$U(t_1,t_2)=\frac{1}{(t_1t_2)}e^{-(N/2)t_1^2-(N/2)t_2^2}\oint\frac{dudv}{(2\pi i)^2}e^{Nit_1u+Nit_2v}\frac{(u-v+(it_1-it_2))(u-v)}{(u-v+it_1)(u-v-it_2)}\prod_{\gamma=1}^{N}\left(1+\frac{it_1}{(u-\epsilon_\gamma)}\right)\left(1+\frac{it_2}{(v-\epsilon_\gamma)}\right). \quad (3.16)$$

Noting that

$$1-\frac{t_1t_2}{(u-v+it_1)(u-v-it_2)}=\frac{(u-v+i(t_1-t_2))(u-v)}{(u-v+it_1)(u-v-it_2)}, \quad (3.17)$$

we find that Eq. (3.16) is a sum of the disconnected term and a connected part. We know Fourier transform U with repect to t_1 and t_2 and shift the integrations variables. By the shifts $t_1\to t_1+iu$, and $t_2\to t_2+iv$, we easily see that $R_2(\lambda,\mu)$ is a 2×2 determinant, namely, that

$$R_2(\lambda,\mu)=K_N(\lambda,\lambda)K_N(\mu,\mu)-K_N(\lambda,\mu)K_N(\mu,\lambda), \quad (3.18)$$

with the kernel

$$K_N(\lambda,\mu)=\int\frac{dt}{2\pi}\oint\frac{dv}{2\pi i}\prod_{\gamma=1}^{N}\left(\frac{\epsilon_\gamma-it}{v-\epsilon_\gamma}\right)\times\frac{1}{v-it}e^{-(N/2)v^2-(N/2)t^2-Nit\lambda+Nv\mu}. \quad (3.19)$$

Note the similarity of the determinantal structure found here with that of the zero source case given in Eq. (2.4).

In [9], this kernel $K_N(\lambda,\mu)$ was examined in the scaling limit, large N, but fixed $N(\lambda-\mu)$. In this limit one can evaluate the kernel (3.19) by the saddle-point method. We have to assume here that the distribution of eigenvalues of H_0 posseses a limit when N goes to infinity. Namely, we assume that

$$\rho_0^{(N)}(\lambda)=\frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\epsilon_i) \quad (3.20)$$

has a finite limit when N goes to infinity. If this assumption is relaxed, it is clear that a different behavior of the correlations could take place; for instance, if the support of the ϵ_i's was growing rapidly with N, one would presumably observe a crossover to a Poissonian regime for the correlations. The result was found to be, up to a phase factor that we omit here,

$$K_N(\lambda_1,\lambda_2)=-\frac{1}{\pi y}\sin[\pi y\rho(\lambda_1)], \quad (3.21)$$

where $y=N(\lambda_1-\lambda_2)$. Apart from the scale dependence provided by the density of state ρ, the two-point correlation function has a universal scaling limit, i.e., indepent of the deterministic part H_0 of the random Hamiltonian.

IV. DETERMINANT FOR THE N-POINT CORRELATION FUNCTION

The n-point correlation function $R_n(\lambda_1,\ldots,\lambda_n)$ is given by

$$R_n(\lambda_1,\ldots,\lambda_n)=\frac{1}{N^n}\left\langle\prod_{i=1}^{n}\mathrm{Tr}\,\delta(\lambda_i-M)\right\rangle. \quad (4.1)$$

If we put the constraints that all λ_i are different, this expression conincides with Eq. (2.1). When some λ_i become the same, we have extra δ functions as shown in [9]. Therefore, we assume all λ_i are different.

Without an external source, this n-point correlation function is expressed in terms of the kernel $K_N(\lambda_i,\lambda_j)$ as [2,4]

$$R_n(\lambda_1,\ldots,\lambda_n)=\det[K_N(\lambda_i,\lambda_j)], \quad (4.2)$$

where $i,j=1,\ldots,n$. This result was derived by the use of the orthogonal polynomials. In the external source problem, we cannot apply the orthogonal polynomial method. Our aim is to find a proof of Eq. (4.2) for the external source case.

Using the Itzykson-Zuber formula of Eq. (3.5), we have

$$R_n(\lambda_1,\cdots,\lambda_n)=\frac{1}{N^n}\sum_{\alpha_i\neq\alpha_j}\int\frac{dt_1\cdots dt_n}{(2\pi)^n}\frac{\Delta(B)}{\Delta(H_0)}\times e^{N/2\Sigma b_i^2+i\Sigma t_k\lambda_k}, \tag{4.3}$$

where

$$b_k=\epsilon_k+i(t_1\delta_{k,\alpha_1}+\cdots+t_n\delta_{k,\alpha_n}). \tag{4.4}$$

Using the contour-integration representation, we get

$$R_n=\int\frac{dt_1\cdots dt_n}{(2\pi)^n}e^{-(N/2)\Sigma t_p^2+iN\Sigma t_p\lambda_p}\oint\frac{du_1\cdots du_n}{(2\pi i)^n}e^{Ni\Sigma t_p u_p}\prod_{p=1}^{n}\prod_{\alpha=1}^{N}\left(1+\frac{it_p}{u_p-\epsilon_\alpha}\right)\prod_p^n\frac{1}{t_p}\prod_{p<q}\frac{[u_p-u_q+i(t_p-t_q)](u_p-u_q)}{(u_p-u_q+it_p)(u_p-u_q-it_q)}. \tag{4.5}$$

When $n=2$, this reduces to the previous expression (3.16). When make a shift of the variables t_p: $t_p\to t_p+iu_p$, then we get

$$R_n=\int\frac{dt_1\cdots dt_n}{(2\pi)^n}\oint\frac{du_1\cdots du_n}{(2\pi i)^n}e^{-(N/2)\Sigma t_p^2-(N/2)\Sigma u_p^2+\Sigma\lambda_p(-iNt_p+Nu_p)}\prod_{p=1}^{n}\prod_{\alpha=1}^{N}\left(\frac{-\epsilon_\alpha+it_p}{u_p-\epsilon_\alpha}\right)$$
$$\times\prod_{p<q}\left(\frac{it_p-it_q}{-u_q+it_p}\right)\frac{(u_p-u_q)}{(u_p-it_q)}\prod_{p=1}^{n}\frac{1}{(t_p+iu_p)}. \tag{4.6}$$

We recognize in Eq. (4.6) a Cauchy determinant,

$$\det\left[\frac{1}{a_i-b_j}\right]_{i,j=1,\ldots,n}=(-1)^{[n(n-1)/2]}\frac{\Pi_{i<j}(a_i-a_j)(b_i-b_j)}{\Pi_{i,j}(a_i-b_j)}, \tag{4.7}$$

if we identify a_k to it_k, and b_k to u_k in Eq. (4.6). Then, R_n is given by

$$R_n=\int\frac{dt_1\cdots dt_n}{(2\pi)^n}\oint\frac{du_1\cdots du_n}{(2\pi i)^n}e^{-(N/2)\Sigma t_k^2-(N/2)\Sigma u_k^2+\Sigma\lambda_k(-iNt_k+Nu_k)}\prod_{k=1}^{n}\prod_{\alpha=1}^{n}\left(\frac{-it_k+\epsilon_\alpha}{\epsilon_\alpha-u_k}\right)\det\left(\frac{1}{it_i-u_j}\right)$$
$$=\int\frac{dt_1\cdots dt_n}{(2\pi)^n}\oint\frac{du_1\cdots du_n}{(2\pi i)^n}e^{-(N/2)\Sigma t_k^2-(N/2)\Sigma u_k^2+\Sigma\lambda_k(-iNt_k+Nu_k)}\det\left[\prod_{\alpha=1}^{N}\frac{-it_i+\epsilon_\alpha}{(it_i-u_j)(\epsilon_\alpha-u_j)}\right]. \tag{4.8}$$

Using the expression for the kernel of Eq. (3.19), we obtain

$$R_n(\lambda_1,\ldots,\lambda_n)=\det[K_N(\lambda_i,\lambda_j)]_{i,j=1,\ldots,n}. \tag{4.9}$$

We could thus prove the determinantal form of the n-point correlation function for a deterministic plus random Hamiltonian.

V. THE PROPERTIES OF THE KERNEL

As we have seen in Eq. (4.9), the n-point correlation function R_n is expressed by the determinant in the presence of the external source. When we integrate out the variables $x_{l+1},\ldots,x_n$ of $R_n(x_1,\ldots,x_n)$, we obtain the one-point correlation function $R_l(x_1,\ldots,x_l)$. Since we have Eq. (4.9), the necessary consistency condition for this result is

$$\int_{-\infty}^{+\infty}d\mu\,K_N(\lambda,\mu)K_N(\mu,\nu)=K_N(\lambda,\nu). \tag{5.1}$$

This property is verified easily by the contour-integral representation of the kernel $K_N(\lambda,\mu)$ given in Eq. (3.19) [10]. We have

$$\int_{-\infty}^{\infty}K_N(\lambda,\mu)K_N(\mu,\nu)d\mu=\int_{-\infty}^{\infty}\frac{dt_1dt_2}{(2\pi)^2}\oint\frac{du_1du_2}{(2\pi i)^2}\prod_\gamma\left(\frac{\epsilon_\gamma+it_1}{u_1-\epsilon_\gamma}\right)\left(\frac{\epsilon_\gamma+it_2}{u_2-\epsilon_\gamma}\right)\frac{1}{(u_1+it_1)(u_2+it_2)}$$
$$\times e^{-(N/2)(u_1^2+u_2^2+t_1^2+t_2^2)-iNt_1\lambda-iNt_2\mu-Nu_1\mu-Nu_2\nu}. \tag{5.2}$$

Integration over μ, after the shift $t_2 \to t_2 + i u_1$, gives a δ function for t_2, and the contour integral over u_1 around the pole $u_1 = -i t_1$ reconstructs $K_N(\lambda,\nu)$.

We also observe the kernel $K_N(\lambda,\mu)$ has N eigenvalues equal to one, with Hermite polynomials as eigenfunctions since, for $n<N$,

$$\int_{-\infty}^{\infty} K_N(\lambda,\mu) H_n(\sqrt{N}\mu) e^{-(N/2)\mu^2} d\mu = H_n(\sqrt{N}\lambda) e^{-(N/2)\lambda^2}, \tag{5.3}$$

with $H_0(x)=1$, $H_1(x)= x$, $H_2(x) = x^2-1$, etc. This property may also be easily verified through the contour-integratal representation. For $n>N-1$, Eq. (5.3) does not hold. The right hand side of Eq. (5.3) becomes ϵ_γ dependent. When the external source ϵ_γ goes to zero, the right hand side of Eq. (5.3) is vanishing for $n>N-1$. This is of course related to the fact that the kernel is then expressed as a finite sum of Hermite polynomials.

VI. SUMMARY AND DISCUSSION

In the preceding section, we have proved that the n-point correlation function is expressed by the kernel $K(x,y)$, as in the absence of an external source. In the short-distance limit, in which $(\lambda-\mu)N$ is kept fixed, the kernel $K_N(\lambda_p,\lambda_q)$ takes a universal form, and the n-point correlation becomes universal (up to a rescaling by the density of state ρ). As we have seen in Sec. II, the level-spacing probability distribution $P(s)$ is given by an integration over the n-point correlation function $R_n(\lambda_i,\ldots,\lambda_n)$. Therefore, in the short-distance scaling limit, $P(s)$ has a universal form, independent of the deterministic part.

We have assumed throughout this work that the density of state is finite, of order one, in the energy range that we are considering. We have discussed the level-spacing probability $P(s)$ for two levels centered around the enregy zero. If we considered instead two levels centered around an energy E_0, i.e., an interval $[-s/2+E_0, s/2+E_0]$, the behavior of the kernel $K_N(\lambda,\mu)$ remains universal, apart from the scaling by the the density of state $\rho(E_0)$instead of $\rho(0)$. Therefore, we still have the same universal spacing dsitribution $P(s)$ for an arbitrary energy E_0 as long as $\rho(E_0)$ remains of order one.

We have also assumed that the eigenvalues of the deterministic term H_0 are inside the support of the asymptotic smooth density of state ρ. When the eigenvalues are widely separated, the density of state shows an oscillatory behavior. In such cases, the two-level correlation function, or the kernel $K_N(\lambda,\mu)$ does not approach, in the scaling limit, the sine kernel, and a universal form for $P(s)$ is not expected [15]. This is reasonable, since we know that when the random potential V increases in comparison with the unperturbed deterministic term H_0, we crossover to a universal behavior independent of the initial deterministic term.

Finally we have found two kinds of universality: either $H_0=0$ and the distribution of H is non-Gaussian, or H_0 is nonzero and V is Gaussian. It is tempting to conjecture that this generalizes to non-Gaussian problems with a nonzero source as considered in the case of the two-level correlation function [11,12,16], or to the time-dependent case [10].

ACKNOWLEDGMENTS

This work was supported by the CNRS-JSPS cooperative project, and by the CREST of JST.

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